J. DIFFERENTIAL GEOMETRY
32 (1990) 155-183

EINSTEIN METRICS

L. ZHIYONG GAO

1. Introduction

One of the natural classes of Riemannian metrics on an n-manifold is Einstein metrics. Berger and Ebin [4] were the first people to study the moduli space of Einstein metrics on an Einstein manifold. They showed that the moduli space of Einstein metrics is finite dimensional. This leads to the natural question whether the moduli space is compact. Einstein metrics have been studied by many mathematicians; there is an excellent book on Einstein manifolds [6].

One of the main accomplishments of this paper is that we found a compactness property of moduli spaces of four-dimensional Einstein manifolds. To explain this, let us start with a 4-manifold M, which has an Einstein metric. We consider the moduli space of all Einstein metrics on M, and normalize the Einstein metrics such that the Ricci curvature equal $+3$, -3, or 0. Let $G(M)$ be the subspace of all normalized Einstein metrics on M with the injective radius bounded from below by a fixed constant $i_0 > 0$ and diameter bounded from above by d. We are able to show that $G(M)$ is compact as a subset of the moduli space of Einstein metrics in C^∞-topology.

Theorem 1.1. *The subset of normalized Einstein metrics with Ricci curvature equal to three and with injectivity radius bounded from below on a 4-manifold M is compact in C^∞-topology.*

Theorem 1.2. *The subset of normalized Einstein metrics with Ricci curvature equal to negative three or zero, with injectivity radius bounded from below and diameter bounded from above on a 4-manifold M is compact in C^∞-topology.*

Remark 1.3. It seems that without the lower bound of the injectivity radius, the results are false.

We briefly describe here the method used in this paper. For a sequence of Einstein manifolds (M_k) with proper restrictions, by passing to a sub-

Received September 11, 1987 and, in revised form, December 19, 1988. The author's research was supported in part by National Science Foundation Grant DMS-87-03345 and the Alfred P. Sloan Foundation.

sequence, we show that (M_k) converges to an Einstein manifold M' almost everywhere except a finite number of points; here M' is an Einstein manifold with a finite number of isolated singular points. We then prove that the singular points can be removed, which, combined with the Gauss-Bonnet formula for a 4-manifold, will imply the theorem. This idea was first used by Sacks and Uhlenbeck [24]. We will combine this idea with Gromov's compactness theorem to prove the convergence of Einstein metrics. The technical key step is to prove that each singular point is isolated and can be removed.

2. The proof

The remainder of this article contains the proofs of the preceding theorems, and it is organized in the following way. In §§3 and 4, we use the equations satisfied by the curvature tensors of Einstein metrics to give the local estimate, which shows that curvature tensor is bounded on the ball with small L^2-norm of curvature.

Next, we use the Gauss-Bonnet formula for a 4-manifold to show that the curvature tensor can be only concentrated on finite number of points; away from these points, the curvatures are uniformly bounded. We then use the argument of [15] or [23] to prove that the Einstein manifolds converge to an Einstein manifold M' away from these points.

Then, by using the diameter estimate of a small geodesic sphere of [12], we show that M' is diffeomorphic to $M - \{m_1, \cdots, m_h\}$, and that these $\{m_i\}$ are isolated singular points of the metric on M'.

Finally, by means of the Gauss-Bonnet formula and the results of §5 about the removable singularities, we first prove that the curvature tensors of the Einstein manifolds are uniformly bounded, and then apply Gromov's compactness theorem to finish our proof.

§5 contains the key technical step. At first a study is given of the *tangent cone metric* of M' at each of these singular points $\{m_i\}$ by blowing up at these points. Then we show that the tangent cone metric is a flat Euclidean metric, and further that the curvature tensor of M' is bounded by integral estimates combined with iteration. These are used to prove that the singular points of M' can be removed.

3. Convergence of Einstein metrics and proof of Theorems 1.1 and 1.2

Let us fix the manifold M^4 and $\sigma = +1, -1$, or 0, and define the set of Riemannian metrics

$$G(M) = \Big\{g | g \text{ a } c^{\infty}\text{-metric on } M \text{ such that } R_{ij}(g) = 3\sigma g_{ij}, \\ \text{inj}(M, g) \geq i_0 > 0, \text{ and } \text{diam}(M, g) \leq d\Big\},$$

where i_0 is a positive number. By Berger's isoembolic inequality, we have $\text{Vol}(M, g) \geq c(i_0)$. From now on, we assume that the general constant c will depend on i_0 and d.

Theorem A. *$G(M)$ is compact in C^{∞}-topology.*

We start with a pointwise estimate of the curvature tensor.

Lemma 3.1. *For any $g \in G(M)$, there exists a $\kappa_5 = \kappa_5(i_0, d) > 0$, such that if*

$$\int_{B(x_0, \rho)} |R_{ijkl}(g)|^2 \, dv_g \leq \kappa_5,$$

and $\rho > 0$, then

$$\sup_{B(x_0, (1-\eta)^{2l}\rho)} |\partial^l R_m(g)| \leq C(\rho, \eta, i_0, d), \qquad l = 0, 1, \cdots, 5.$$

For the proof of this lemma, see Theorem 4.11 in the next section.

Using this lemma, we first prove a result on the weak convergence.

Proposition 3.2. *Let $\{g^k\} \subset G(M)$, $k = 1, 2, \cdots$, be a sequence in $G(M)$, $M^k = (M, g^k)$. Then there exists a subsequence of $\{g^k\}$ (by renumbering but still using $\{g^k\}$) and a sequence $\{r_l\}$, $r_l \to 0$.*

(a) *For each l, we have open subsets $F_k(r_l) \subset M^k$, and an open subset $D(r_l) \subset M$.*

(b) *For each l, we have diffeomorphisms*

$$f^k(r_l): D(r_l) \to F_k(r_l) \qquad (k \geq l), \cdots$$

such that $f^k(r_l)^* g^k$ converges to a C^{∞}-Einstein metric $g(r_l)$ on $D(r_l)$ in C^2-norm.

(c) *$F_k(r_l) \subset F_k(r_{l+1})$.*

(d) *There is an $\varepsilon(r_l)$, such that*

$$F_k(r_l) \cup \bigcup_{i=1}^{N} B^k(x^k_{i,l}, \varepsilon(r_l)) = M^k,$$

and $\varepsilon(r_l) \to 0$ when $r_l \to 0$ with $N \leq c(i_0, \chi(M))$, where $\chi(M)$ is the Euler number of M.

We employ the covering argument of [24], and combine it with the argument of [23] or [15].

Proof. Let $r < i_0/100$. Given a sequence $\{g^k\}$ in $G(M)$, let $Q(k)$ be the maximum number of disjoint geodesic balls of radius $r/3$. By the Gromov packing argument, since the diameter $d(M, g^k)$ of $(M, g^k) \leq d$, and $\mathrm{Ric} \geq -3$, we have

$$Q(k) \leq c(r). \tag{3.3}$$

By passing to a subsequence if necessary, we assume that $Q(k) \equiv Q \in Z^+$ for all k.

We now fix k, and let $\{B^k(x_i, r/3)\}$, $i = 1, \cdots, Q$, be a maximal family of disjoint geodesic balls of radius $r/3$. Then $\{B^k(x_i, r)\}$, $i = 1, \cdots, Q$, is a covering of $(M, g^k) = M^k$. Let $h(k)$ be the maximal number such that any $h(k) + 1$ balls of $\{B^k(x_i, r)\}$ have empty intersection. Let

$$\{B^k(x_{i\alpha}, r)\}, \qquad \alpha = 1, \cdots, h(k),$$

be the balls of $\{B^k(x_i, r)\}$ such that

$$B^k(x_{i_1}, r) \cap \cdots \cap B^k(x_{i_h}, r) \neq \emptyset.$$

Then $B^i(x_{i_\alpha}, r) \subset B^k(x_{i_h}, 3r)$, $\alpha = 1, \cdots, h$. Since $\{B^i(x_{i_\alpha}, r/3)\}$ are disjoint geodesic balls, by the Bishop-Gromov volume estimate, we have

$$h(k) = \max_\alpha \frac{\mathrm{Vol}(B^k(x_{i_h}, 3r))}{\mathrm{Vol}(B^k(x_{i_\alpha}, r/3))} \leq \max_\alpha \frac{\mathrm{Vol}(B^k(x_{i_\alpha}, 5r))}{\mathrm{Vol}(B^k(x_{i_\alpha}, r/3))} \leq c. \tag{3.4}$$

So the $h(k)$ are uniformly bounded by c. We would like to apply the estimates of Lemma 3.1 to each ball of $\{B^k(x_i, r)\}$, but the hypothesis need not be met on all balls. However, we have an upper bound on the number of balls on which it fails; by the Gauss-Bonnet formula of a 4-dimensional Einstein manifold [5] we have

$$\begin{aligned} 8\pi^2 h(k)\chi(M) &\geq h \int_M |R_{ijkl}(gk)|^2 \, dv_{gk} \geq \sum_{i=1}^{Q} \int_{B^k(x_i, r)} |R_{ijkl}(gk)|^2 \, dv_{gk} \\ &\geq \sum_{i=1}^{Q} \int_{B^k(x_i, r)} |R_{ijkl}(gk)|^2 \, dv_{gk} \geq N(k)\kappa_5, \end{aligned}$$

where $N(k)$ is the number of $B^k(x_i, r)$ for which

$$\int_{B^k(x_i, r)} |R_{ijkl}(gk)|^2 \, dv_{gk} \geq \kappa_5.$$

Thus

$$N(k) \leq c(i_0, \chi(M), d), \tag{3.5}$$

which means that $N(k)$ are uniformly bounded by a constant $c(d, i_0, \chi(M))$. By passing to a subsequence if necessary, we assume that $N(k)$ and $h(k)$ are constant, i.e., $N(k) \equiv N \in Z^+$, $h(k) \equiv h \in Z^+$ for all k and $r < i_0/100$.

Remark. We will call a ball $B^k(x_i, r)$ a bad ball for M_k if

$$\int_{B^k(x_i, r)} |R_{ijk\rho}(gk)|^2 \, dv \geq \kappa_5.$$

Otherwise, it is called a good ball.

Let $Q' = Q - N$, and denote the good balls by

$$\{B^k(x_i, r)\}, \qquad i = 1, 2, \cdots, Q',$$

and the bad balls by

$$\{B^k(x_i, r)\}, \qquad i = Q' + 1, \cdots, Q.$$

Let

$$U_k(r) = \bigcup_{i=1}^{Q'} B^k(x_i, (1-\eta)^{14} r)$$

for small $\eta > 0$. By Lemma 3.1, we have

$$\sup_{U_k(r)} |\partial^l R_m(gk)| \leq c(r), \qquad l = 0, 1, \cdots, 5. \tag{3.6}$$

Assuming $(1-\eta)^{14} > 1 - \frac{1}{100} = \frac{99}{100}$, it follows that

$$\{B^k(x_i, (1-\eta)^{14} r)\}, \qquad i = 1, \cdots, Q,$$

is a covering of M_k, so

$$U_k(r) \cup \bigcup_{i=Q'+1}^{Q} B^k(x_i, r) = M.$$

We take $\bar{r} = \min\{r/300, i_0\} = (\eta_1) r$. Letting

$$V_k(r) = \bigcup_{j=1}^{Q'} B^k(x_j, ((1-\eta)^{14} - \eta_1) r),$$

we have

$$\bigcup_{j=1}^{Q} B^k(x_j, ((1-\eta)^{14} - \eta_1)r) = M, \tag{3.7}$$

$$V_k(r) \cup \bigcup_{j=Q'+1}^{Q} B^k(x_j, r) = M.$$

By taking even smaller balls in $U_k(r)$ to cover $V_k(r)$, as in [23], for $P(r)$ large, there exist an open set $D(r) \supset V_{P(r)}(r)$, subsets $F_k(r)$, and diffeomorphisms

$$f^k(r): F_k(r) \mapsto D(r)$$

such that

$$V_k(r) \subset F_k(r) \subset U_k(r).$$

Then we have

Lemma 3.8. *There exists a C^∞-Einstein metric $g(r)$ on $D(r) \supseteq V_{P(r)}(r)$ such that $\{(f^k)^* gk\}$ (by passing to a subsequence if necessary) converges to $g(r)$ on $D(r)$ in C^∞-norm.*

Let $r_1 = i_0/1000$ and $r_2 = r_1/8000$. By renumbering if necessary, we assume that $\{B^k(x_{i,2}^k, r_2)\}$, $i = 1, \cdots, Q'(r_2)$, are good balls, and that $\{B^k(x_{i,2}^k, r/2)\}$, $i = Q'(r_2)+1, \cdots, Q(r_2)$, are bad balls. Then we have

$$\int_{B^k(xl_i^2, r_2)} |R_{ijkl}|^2 d^k < \kappa_5, \quad \text{for } i = 1, 2, \cdots, Q'(r_2),$$

$$\int_{B^k(x_i^2, r_2)} |R_{ijkl}|^2 d^k \geq \kappa_5 \quad \text{for } i = Q'(r_2)+1, \cdots, Q'(r_2),$$

and $Q(r_2) - Q'(r_2) \leq c(i_0, \chi(M))$.

We recall that $\{B^k(x_i, r_1)\}$, $i = 1, \cdots, Q'(r_1)$, are good balls, $\{B^k(x_i, r_1)\}$, $i = Q'(r_1)+1, \cdots, Q(r_1)$, are bad balls, and

$$\bigcup_{i=1}^{Q(r_1)} B^k(x_i, (1-\eta)^{14} r_1) = M.$$

Lemma 3.9. $V_k(r_2) \supset U_k(r_1)$.

Proof. We have

$$V_k(r_2) \cup \bigcup_{j=Q'(r_2)+1}^{Q(r_2)} B^k(x_{j,2}^k, ((1-\eta)^{14} - \eta_1)r_2) = M,$$

$$U_k(r_1) \cup \bigcup_{j=Q'(r_1)+1}^{Q(r_1)} B^k(x_j, (1-\eta)^{14} r_1) = M.$$

By the definition of bad balls, we have

$$U_k(r_1) \cap B^k(x_{j,2}^k, r_2) = \varnothing, \qquad j = Q'(r_2)+1, \cdots, Q'(r_2);$$

otherwise

$$B^k(x_{j,2}^k, r_2) \cap B^k(x_i, (1-\eta)^{14} r_1) \neq \varnothing$$

for some $i = 1, \cdots, Q'(r_1)$, and thus

$$B^k(x_{j,2}^k, r_2) \subset B^k(x_i, r_1), \qquad i \leq Q'(r_1),$$

$$\int_{B^k(x_{j,2}^k, r_2)} |R_{ijkl}|^2 \, dv^k \leq \int_{B^k(x_i, r_1)} |R_{ijkl}|^2 \, dv^k < \kappa_5,$$

which means that $B^k(x_{j,2}^k, r_2)$, $Q'(r_2)+1 \leq j \leq Q(r_2)$, is a good ball. Therefore, $V_k(r_2) \supset U_k(r_1)$, and

$$B^k(x_{j,2}^k, r_2) \subset B^k(x_i, r_1), \qquad j > Q'(r_2),$$

for some $i \geq Q'(r_1) + 1$.

By induction on l, we take $r_{l+1} = \frac{1}{8000} r_l$.

By diagonalization (passing to a subsequence if necessary) and summarization we have

$$F_k(r_{k+1}) \supset V_k(r_{l+1}) \supset F_k(r_l),$$

$$F_k(r_l) \cup \bigcup_{i=1}^{N} B^k(x_{i,l}^k, \varepsilon(r_l)) = M,$$

where $\varepsilon(r_l) \to 0$ when $r_l \to 0$. This completes the proof of Proposition 3.2.

Next, we use these Einstein manifolds $\{D(r_l)\}$ to construct an Einstein manifold M' with isolated singularities.

Proposition 3.10. *There exists an Einstein manifold M' such that each $D(r_l)$ is an embedded submanifold of M' with the induced metric $g(r_l)$ on $D(r_l)$. We identify $D(r_l)$ with its embedded image, and let g' be the metric* *on M'. Then for each l, there are diffeomorphisms $f^k(r_l)\colon F_k(r_l) \to D(r_l) \subset M'$ for $k \geq P(r_l)$ such that $f^k(r_l)^* gl^k \to g'|_{D(r_l)}$ in C^2-norm.*

Proof. By 3.2 there exist a subsequence of $\{g^k\}$ (by renumbering if necessary), still called $\{g^k\}$, and open subsets $F_k(r_l) \subset M^k$, $D(r_l) \subset M$, such that $r_l \to 0$, $\overline{D(r_l)}$ and $\overline{F_k(r_l)}$ are compact, and $F_k(r_l) \subset F_l(r_{l+1})$. We also have diffeomorphisms $f^k(r_l) = D(r_l) \to F_k(r_l)$ for $k \geq P(r_l)$ such that

$$(f^k(r_l))^* g^k \xrightarrow{c^2} g(r_l) \quad \text{on } D(r_l)$$

for a C^∞-Einstein metric $g(r_l)$ on $D(r_l)$.

First we will define an isometric embedding

$$I_l\colon (D(r_l), g(r_l)) \to (D(r_{l+1}), g(r_{l+1})).$$

In fact, by the construction of the $F_k(r_l)$'s we have

$$F_k(r_l) \subset U_k(r_l) \subset V_k(r_{l+1}) \subset F_k(r_{l+1}),$$
$$\overline{\bigcup_{\bar{r}_l}(F_k(r_l))} \subset F_k(r_{l+1}),$$

where $\overline{\bigcup_{\bar{r}_l}(F_k(r_l))}$ is compact, and $d^k(F_k(r_l)), M - F_k(r_{l+1}) \geq \bar{r}_l$.

Since $f^k(r_l)^* g^k$ converges to $g(r_l)$ on $D(r_l)$ in C^2-norm for each l, for any given $\varepsilon > 0$ if k is sufficiently large, we have for the C^0-norms,

$$|f^k(r_l)^* g^k - g(r_l)|_{f^k(r_l)^* g^k} \leq \varepsilon,$$
$$|f^k(r_{l+1})^* g^k - g(r_{l+1})|_{f^k(r_l)^* g^k} \leq \varepsilon,$$

which clearly imply

$$|(f^k(r_l)^{-1})^* g(r_l) - g^k|_{g^k} \leq \varepsilon,$$
$$|(f^k(r_{l+1})^{-1})^* g(r_{l+1}) - g^k|_{g^k} \leq \varepsilon.$$

Therefore, we have

$$|(f^k(r_l)^{-1})^* g(r_l) - (f^k(r_{l+1})^{-1})^* g(r_{l+1})|_{g^k} \leq 2\varepsilon,$$

and hence

$$|(f^k(r_{l+1})^{-1} \circ f^k(r_l))^* g(r_{l+1}) - g(r_l)|_{g(r_l)} \leq c\varepsilon,$$

which shows that

$$\lim_{k\to\infty} (f^k(r_{l+1})^{-1} \circ f^k(r_l))^* g(r_{l+1}) = g(r_l).$$

Let

$$I^k(r_l) = f^k(r_{l+1})^{-1} \circ f^k(r_l)\colon D(r_l) \to D(r_{l+1}).$$

We consider $(D(r_l), g(r_l))$ and $(D(r_{l+1}), g(r_{l+1}))$ as metric spaces with induced metrics $d^k(l)$ and $d^k(r_{l+1})$ on $D(r_l)$ and $D(r_{l+1})$. For k large, we have

$$d^k(r_{l+1})(I^k(r_l)(x), I^k(r_l)(y)) \le 2d^k(l)(x, y).$$

Note that $d^k(I^k(r_l)(D(r_l)), M - D(r_{l+1})) \ge r_l$ implies that $\overline{I^k(r_l)(D(r_l))} \subset D(r_{l+1})$, and that $I^k(r_l)(D(r_l))$ is precompact in $D(r_{l+1})$. By the Ascoli theorem [21], $I^k(r_l)$ (by passing to a subsequence if necessary) converges to a continuous map I_l on $D(r_l)$, and I_l is a Lipschitz map. Since

$$\lim_{k\to\infty} I^k(r_l)^* g(r_{l+1}) = g(r_l),$$

this implies that

$$I_l\colon (D(r_l), d(r_l)) \to (D(r_{l+1}), d(r_{l+1}))$$

is an isometry of metric spaces on small convex geodesic balls of $(D(r_l), g(r_l))$. Thus it follows that

$$I_l\colon (D(r_l), g(r_l)) \to (D(r_{l+1}), g(r_{l+1}))$$

is a C^∞-isometry of a Riemannian manifold [20], i.e., $I_l^* g(r_{l+1}) = g(r_l)$.

We now can construct the Riemannian manifold (M', g'). We define M' to be the direct limit of $\{D(r_l)\colon I_l\colon D(r_l) \to D(r_{l+1}),\ l = 1, 2, \cdots\}$. Letting

$$Y = \amalg_{l=1}^{\infty} D(r_l),$$

we identify the point x, y in Y if $y = I_l(x)$ for some l. The quotient space is defined to be M'. Then M' is a 4-manifold, and each $D(r_l)$ is an open submanifold of M' with the embedding $J_l = D(r_l) \to Y \to M'$. We define the Riemannian metric g' on M' by $g'|D(r_l) = g(r_l)$, and g' is well defined since $I_l^* g^k(r_{l+1}) = g(r_l)$.

Summarizing, for each l there are diffeomorphisms $f^k(r_l)\colon D(r_l) \to F_k(r_l) \subset M^k$, such that

$$f^k(r_l)^* g^k \xrightarrow{c^2} g'|_{D(r_l)} \quad \text{on } D(r_l) \subset M',$$

which implies that g' is a weak c^2-limit of g^k on M. Hence the proof of Proposition 3.10 is complete.

Next we will study the topology of M'. First let us recall Theorem 2.30 from [13] (see 4.14 in the next section).

Theorem 3.11. *Let (M, g) be a complete Riemannian manifold with $\operatorname{inj}(M, g) \geq i_0 > 0$ and $\operatorname{Ric}(g) \geq -(n-1)g$. Then the diameter of the small geodesic sphere $S_r(x) = \{y \in M | d(x, y) = r\}$ for $x \in M$ and $r \leq i_0/4$ satisfies:*

$$\operatorname{diam}(S_r(x)) \leq c(n, i_0)r,$$

where the constant $c(n, i_0)$ depends only on n and i_0.

Remark. In 3.11 we consider $S_r(x)$ as a submanifold of (M, g) with induced Riemannian metric; the diameter is with respect to the induced distance of this induced Riemannian metric.

We now have

$$F_k(r_l) \cup \bigcup_{j=1}^{N} B(m_j, \varepsilon(r_l)) = M.$$

From 3.11, we note that the injectivity radius of M^k is bounded from below by $i_0 > 0$ for r_l small, and that each $B(m_j, \varepsilon(r_l))$ is contained in a diffeomorphism geodesic ball. These clearly imply the following theorem.

Theorem 3.12. *M' is diffeomorphic to $M - \{v_1, v_2, \cdots, v_{s+1}\}$ for a finite number of points $\{v_1, v_2, \cdots, v_{s+1}\}$ in M, and each v_j is an isolated singularity of the metric g'.*

In the next section we will prove

Theorem 3.13. *Each singularity v_i of g' can be removed, and g' can be extended to a C^∞-Einstein metric g on M.*

Proof of Theorem A. Let $g^k \in G(M)$. We may assume that $\{g^k\}$ converges weakly in c^2-norm to a C^∞-Einstein metric g' on $M' = M - \{v_1, \cdots, v_{s+1}\}$ as in 3.10. Using Theorem 3.13, g' can be extended to an Einstein metric g on M. Using formulas

$$\chi(M) = \frac{1}{8\pi^2}\int_M |R(g)|^2\, dg = \frac{1}{8\pi^2}\int_{M'} |R(g')|\, dg,$$
$$\chi(M) = \frac{1}{8\pi^2}\int_M |R(g^k)|^2\, dg^k,$$

we obtain

$$\bigcup_{l=1}^{\infty} D(r_l) = M',$$

which implies

$$\lim_{l\to\infty}\int_{M'-D(r_l)} |R(g)|^2\, dg = 0. \tag{3.14}$$

Recalling from above, we have

$$U_k(r_l) \cup \bigcup_{j=Q'(r_l)+1}^{Q(r_l)} B^k\left(x^k_{j,l}, \frac{199}{200} r_l\right) = M,$$

and $F^k(r_l) \subset U_k(r_l) \subset F^k(r_{l+1}) \subset U_k(r_{l+1})$. From the proof of Lemma 3.9 it also follows that

$$U_k(r_l) \cap \bigcup_{j=Q'(r_{l+1})+1}^{Q(r_{l+1})} B^k(x^k_{j,l+1}, r_{l+1}) = \varnothing. \tag{3.15}$$

Taking l_0 large, such that if $l \geq l_0$, we have

$$\int_{M-D(r_l)} |R(g)|^2 \, dg < \frac{1}{2}\kappa_5. \tag{3.16}$$

On the other hand, since $f^k(r_l)^*(g^k)$ converges to $g|_{D(r_l)}$ in C^2-norm for each l, there is a large $k_0 > 0$, such that if $k \geq k_0$, then

$$\left| \int_{D(r_{l_0})} |R(g)|^2 \, dg - \int_{D(r_{l_0})} f^k(r_{l_0})^*(|R(g^k)|^2 \, dg^k) \right| < \frac{1}{2}\kappa_5. \tag{3.17}$$

(3.16) and (3.17) imply

$$\begin{aligned}
\int_{M-F^k(r_{l_0})} |R(g^k)|^2 \, dg^k &= 8\pi^2\chi(M) - \int_{F^k(r_{l_0})} |R(g^k)|^2 \, dg^k \\
&= 8\pi^2\chi(M) - \int_{D(r_{l_0})} f^k(r_{l_0})^*(|R(g^k)|^2 \, dg^k) \\
&\leq 8\pi^2\chi(M) - \int_{D(r_{l_0})} |R(g)|^2 \, dg + \frac{1}{2}\kappa_5 \\
&\leq \int_{M-D(r_{l_0})} |R(g)|^2 \, dg + \frac{1}{2}\kappa_5 < \kappa_5.
\end{aligned}$$

Combining this with (3.15) yields

$$\int_{B^k(x^k_{j,l_1}, r_{l_1})} |R(g^k)|^2 \, dg^k < \kappa_5, \qquad j = Q'(r_{l_1}) + 1, \cdots, Q(r_{l_1}),$$

where $l_1 = l_0 + 1$. This clearly implies that for r_{l_1}, there are no *bad balls* and $U_k(r_{l_1}) = M$ for $k > k_0$; 3.1 gives us

$$\sup_{U_k(r_{l_1})} |\partial^p R(g^k)| \leq C(r_{l_1}, i_0, \chi(M)), \qquad p = 0, 1, \cdots, 5.$$

We then obtain

$$\sup_M |\partial^p R(g^k)| \le C(r_{l_1}, d, i_0, \chi(M)), \qquad p = 0, 1, \cdots, 5.$$

Using the Gromov convergence theorem [15] we may assume that $\{G^k\}$ (passing to a subsequence if necessary) converges to an Einstein metric $g \in G(M)$ in C^2-norm. The standard elliptic theory then implies that the convergence is in C^∞-norm. As a consequence of the proof, we have the following.

Corollary 3.18. *There exists a constant* $c(i_0, \chi(M), d) > 0$ *such that for all* $g \in G(M)$,

$$\sup_M |R(g)| \le C(i_0, \chi(M), d).$$

4. The equation of an Einstein metric and local estimates of curvature

We will use the old-fashioned index notation for tensors as in [17]. We consider an Einstein metric g_{ij} on a compact 4-manifold M, i.e.,

$$R_{ij} = 3\sigma g_{ij}, \qquad \sigma = +1, -1, \text{ or } 0.$$

We define

$$\Delta R_{ij} = g^{pq}\partial_p\partial_q R_{ijkl}.$$

We need a formula for ΔR_{ijkl}. To this end, we consider the tensors

$$B_{ijkl} = g^{pr}g^{qs}R_{piqj}R_{rksl}$$

as in [17].

We have the following.

Lemma 4.1 [16, *Lemma* 7.2]. *For any metric* g_{ij}, *the curvature tensor* R_{ijkl} *satisfies the identity*

$$\begin{aligned}\Delta R_{ijkl} &+ 2(B_{ijkl} - B_{ijkl} - B_{iljk} + B_{ikjl}) \\ &= \partial_i\partial_k R_{jl} - \partial_i\partial_l R_{jk} - \partial_j\partial_k R_{il} + \partial_j\partial_l R_{ik} + g^{pq}(R_{pjkl}R_{qi} + R_{ipkl}R_{qj}).\end{aligned}$$

If g_{ij} is an Einstein metric, we have $\partial_k R_{ij} = 3\sigma\partial_k g_{ij} \equiv 0$. This gives the following.

Lemma 4.2. *For an Einstein metric* g_{ij}, *the curvature tensor* R_{ijkl} *satisfies the equation*

$$\Delta R_{ijkl} + 2(B_{ijkl} - B_{ijlk} - B_{iljk} + B_{ikjl}) = 6R_{ijkl}.$$

As a corollary we have

Corollary 4.3. *There is a constant* $c > 0$ *such that*

$$-c|R_{ijkl}|^3 + 6|R_{ijkl}|^2 \leq \langle \Delta R_{ijkl}, R_{ijkl}\rangle \leq c|R_{ijkl}|^3 + 6|R_{ijkl}|^2.$$

By differentiating the equation in 3.2, and using Ricci identity, we easily obtain the following lemma.

Lemma 4.4. *There is a constant* $c > 0$ *such that*

$$|\Delta(\partial_r R_{ijkl})| \leq C|\partial_r R_{ijkl}|\,|R_{ijkl}| + 6|\partial_r R_{ijkl}|.$$

In this section, following [28], we will derive the local curvature estimate. Let $V(M)$ be the volume of the Riemannian (M, g), and $d(M)$ be the diameter of (M, g). Throughout this section c will denote a general constant depending only on the lower bound V of $V(M)$ and the upper bound d of $d(M)$, and $c(x, y)$ the general constant which depends only on V, d, and x, y, etc.

Theorem 4.5. *If the Ricci curvature* $\mathrm{Ric}(g) \geq -3$ *for a Riemannian metric* g *on the* 4*-manifold* M^4, *then for any* $f \in H_1^2(M)$,

$$\|f\|_4^2 \leq c_1(\|\nabla f\|_2^2 + \|f\|_2^2).$$

Theorem 4.6. *If the Ricci curvature* $\mathrm{Ric}(g) \geq -3$ *for a Riemannian metric* g *on the* 4*-manifold* M^4, *then for any* $f \in H_1^q(M)$, $1 \leq q < 4$, *we have the Sobolev inequality*

$$\|f\|_p \leq c_2(\|\nabla f\|_q + \|f\|_q),$$

where $1/p = 1/q - 1/4$.

Let $d\mu$ be the volume element of g; then $V(M) = \int_M d\mu$. Since M is a 4-manifold with Einstein metric g, i.e., $R_{ij} = 3\sigma g_{ij}$, we have the following formula for the Euler characteristics [5];

$$\chi(M) = \frac{1}{8\pi^2}\int_M \mathrm{tr}\, R^2\, d\mu_g = \frac{1}{8\pi^2}\int_M |R_{ijkl}|^2\, d\mu_g > 0.$$

Let $B(x_0, \rho) = \{x \in M \mid d(x, x_0) < p\}$ be the geodesic ball in M with center $x_0 \in M$ and radius $0 < \rho \leq \pi/2$.

Lemma 4.7. *Let* $f \in C^2(B(x_0, \rho))$ *and* $-\Delta f \leq bf + A$, $f \geq 0$, $b \geq 0$, *for a constant* $A \geq 0$. *For any* $\eta > 0$, *if*

$$\int_{B(x_0, \rho)} b^2\, d\mu < \frac{1}{4c_1^2},$$

then there is a constant

$$c\left(A, \rho, \eta, \int_{B(x_0, \rho)} f^2\, d\mu\right) > 0$$

such that

$$\sup_{B(x_0, (1-\eta)\rho)} f \le c\left(A, \rho, \eta, \int_{B(x_0, \rho)} f^2 \, d\mu\right).$$

Proof. The standard Moser iteration easily yields 4.7 [28].

In the following, the constants $\kappa_1, \kappa_2, \cdots$ depend only on the lower bound V of $V(M)$, and the upper bound d of $d(M)$.

Theorem 4.8. *Let $\kappa_1 > 0$ be small. Then for any $0 < \eta < 1$, there is a constant $c(\rho, \eta)$, such that if*

$$\int_{B(x_0, \rho)} |R_{ijkl}|^2 \, d\mu < \kappa_1,$$

then

$$\sup_{B(x_0, (1-\eta)^2\rho)} |R_{ijkl}| \le c\left(\rho, \eta, \int_{B(x_0, \rho)} |R_{ijkl}|^2 \, d\mu\right).$$

Proof. From Corollary 4.3, taking $f = |R_{ijkl}|$, we have

$$\Delta f = \frac{\langle \partial R, R\rangle + |\nabla R|^2}{f} - \frac{\langle \nabla R, R\rangle^2}{f^3} \ge -(cf^2) - 6f \ge -(c+2)f^2.$$

Taking $b = (c+2)f$ and $A = 0$ gives

$$\int_{B(x_0, \rho)} b^2 \, d\mu \le \frac{1}{4c_1^2}.$$

For small κ_1, 4.8 follows easily from 4.7.

Theorem 4.9. *Let $\kappa_2 > 0$ be small. Then for any $0 < \eta < 1$ and*

$$\int_{B(x_0, \rho)} |R_{ijkl}|^2 \, d\mu \le \kappa_2,$$

we have

$$\sum_{B(x_0, (1-\eta)^4\rho)} |\partial_p R_{ijkl}| \le c(\rho, \eta).$$

Proof. This easily follows from 4.4 and 4.7.

Lemma 4.10. *For each positive integer $m \ge 1$, there is an absolute constant A_m such that*

$$|\Delta(\partial^m R)| \le A_m \left[\left(\sum_{l \le m-1} |\partial^l R|\right)^2 + |\partial^m R|\,|R|\right].$$

Proof. Using the equation for ΔR_{ijkl}, taking covariant derivative m times, and using Ricci identities, by Lemma 3.1 it is clear that such a constant A_m exists.

Theorem 4.11. *Let $\kappa_m > 0$ be small, $m = 3, 4, \cdots$. Then for $0 < \eta < 1$ and $\rho > 0$,*

$$\int_{B(x_0, \rho)} |R_{ijkl}|^2 \, d\mu \le \kappa_m$$

implies that

$$\sup_{B(x_0, (1-\eta)^{2m}\rho)} |\partial^m R| \le C(m, \rho, \eta), \qquad m = 0, 1, 2, \cdots. \tag{4.12}$$

Proof. We prove this theorem by induction. Theorems 4.8 and 4.9 imply that (4.11) is true for $m = 0$ and $m = 1$. Assuming that Theorem 4.11 is true for m, by Lemma 4.10 we have

$$\begin{aligned}\Delta|\partial^{m+1} R| &\ge \frac{\langle \Delta(\partial^{m+1} R), \partial^{m+1} R}{|\partial^{m+1} R|} \\ &\ge -A_{m+1}\left[\left(\sum_{l \le m} |\partial^l R|\right)^2 + |\partial^{m+1} R|\,|R|\right].\end{aligned}$$

The induction assumption then gives us

$$\Delta|\partial^{m+1} R| \ge -A_{m+1} C(\rho, \eta, m) - A_{m+1}|R|\,|\partial^{m+1} R| \quad \text{on } B(x_0, (1-\eta)^{2m}\rho). \tag{4.13}$$

Taking $b = A_{m+1}|R|$, $f = |\partial^{m+1} R|$, $A = A_{m+1} C$, and using Lemma 4.7, we obtain

$$\sup_{B(x_0, (1-\eta)^{2m+2}\rho)} f \le c\left(\rho, \eta, m, \int_{B(x_0, (1-\eta)^{2m+1}\rho)} f^2 \, d\mu\right).$$

Taking a function u on $B(x_0, (1-\eta)^{2m}\rho)$ such that $u \equiv 1$ on $B(x_0, (1-\eta)^{2m+1}\rho)$, $\operatorname{supp} u \subset B(x_0, (1-\eta)^{2m}\rho)$, and $|\nabla u| \le 2/\eta(1-\eta)^{2m}\rho$, we have

$$\begin{aligned}\int_{B(x_0, (1-\eta)^{2m}\rho)} u^2 f^2 \, d\mu = &- \int_{B(x_0, (1-\eta)^{2m}\rho)} u^2 \langle \Delta(\partial^m R), \partial^m R\rangle \, d\mu \\ &- 2\int_{B(x_0, (1-\eta)^{2m}\rho)} u \nabla u \cdot \langle \nabla(\partial^m R), \partial^m R\rangle \, d\mu.\end{aligned}$$

Using (4.10) and the induction assumption then implies

$$\begin{aligned}\int u^2 f^2 \, d\mu &\le c(m, \rho, \eta) + c(m, \rho, \eta) \int_{B(x_0, (1-\eta)^{2m}\rho)} u|\nabla(\partial^m R)| \, d\mu \\ &\le c(m, \rho, \eta) + \frac{1}{2}\int u^2 f^2 \, d\mu + c(m, \rho, \eta),\end{aligned}$$

where the last inequality is obtained by applying the Cauchy inequality $2ab \le \delta a^2 + \frac{1}{\delta} b^2$. From the above formula it follows that

$$\int_{B(x_0, (1-\eta)^{2m+1}\rho)} f^2 \, d\mu \le \int u^2 f^2 \, d\mu \le c(m, \rho, \eta),$$

which thus implies Theorem 4.11.

Now for the sake of the reader, we will sketch the proof of Theorem 3.11.

Theorem 4.14. *For $H \ge 0$ and $i_0 > 0$ there exists a constant $c = c(H, i_0, \eta) > 0$ such that for any n-dimensional Riemannian manifold (M, g) with $\operatorname{Ric}(g) \ge -H$, $\operatorname{inj}(M, g) \ge i_0 > 0$ and $r \le \frac{1}{2} i_0$, we have*

$$\operatorname{diam}(S_r(x_0)) \le cr, \qquad x_0 \in M.$$

Proof. For $r \le \frac{1}{2} i_0$, we may rescale the metric $\overline{g} = g/r^2$. Since for this new metric $\overline{g}$ we have $\operatorname{Ric}(\overline{g}) \ge -r^2 H \ge -(\frac{1}{2} i_0)^2 H$ and $\operatorname{inj}(M, \overline{g}) \ge 2$, we need only to show

(a) $\operatorname{diam}(S_1(x_0)) \le c(H, i_0, n)$ for this new metric. So we may assume that $i_0 \ge 2$.

To this end, we use a geodesic polar coordinate $\{r, x^1, \cdots, x^{n-1}\}$ on $B_2(x_0)$ to obtain

$$g = dr^2 + \sum g_{ij}(x, r) \, dx^i \, dx^j.$$

Then the Ricci curvature is given by

$$R_{rr} = -\frac{\partial^2}{\partial r^2} \ln \sqrt{g} - \frac{1}{4} \sum g^{ij} g^{kl} \frac{\partial}{\partial r} g_{ik} \frac{\partial}{\partial r} g_{jl} = -\frac{\partial^2}{\partial r^2} \ln \sqrt{g} - \frac{1}{4} \left| \frac{\partial}{\partial r} g_{ij} \right|_g^2 .$$

Using the cut-off function, integrating by parts, and noting that

$$\left| \frac{\partial}{\partial r} \ln g \right|^2 \le (n-1) \left| \frac{\partial}{\partial r} g_{ij} \right|_g^2 ,$$

we easily obtain, for $\rho \le \frac{3}{2}$,

$$\int_0^\rho r^2 \left| \frac{\partial}{\partial r} g_{ij} \right|_g^2 dr \le c(H, i_0, n) \rho$$

which readily gives

(b)

$$e^{-cr_2/r_1} g_{ij}(x, r_1) \le g_{ij}(x, r_2) \le e^{cr_2/r_1} g_{ij}(x, r_1)$$

(for details see §I of [13]).

Now for any sequence of Riemannian manifolds (M_l, g_l) with $\mathrm{Ric}(g_l) \geq -H$ and $\mathrm{inj}(g_l) \geq 2$, we fix $x_l \in M_l$, and consider the ball $B_4(x_l) \subset M_l$ with the extrinsic distance. Then there exists a compact metric space Z [15], such that $B_4(x_l)$ is isometrically embedded in Z. By passing to a subsequence if necessary, we can assume that $x_l \to x_0 \in Z$, and $\{B_4(x_l)\}$ converges to $S_1^X(x_0) \subset X$ in Haussdorff distance of Z. Since X is a length space and $S_1^X(x_0)$ is connected, for any two points $y, z \in S_1^X(x_0)$, there is a sequence of points y_j, $j = 0, 1, \cdots, m$, such that $y_0 = y$, $y_m = z$, and $d(y_j, y_{j+1}) < \frac{1}{5}$. Now for y^l, $z^l \in S_1(x_l)$ such that

$$\mathrm{diam}(S_1(x_l)) = \inf_{\gamma \subset S_1(x_l)} \{L(\gamma) \colon \gamma(0) = y^l, \ \gamma(1) = z^l\},$$

we may assume that

$$y^l \to y, \quad z^l \to z, \qquad y, z \in S_1^X(x_0).$$

If there exist

$$y_j, \quad j = 0, 1, \cdots, m, \qquad d(y_j, y_{j+1}) < \tfrac{1}{5}, \quad y_j \in S_1^X(x_0),$$

then there is a sequence of points

$$\{y_j^l, \ j = 0, 1, \cdots, m\}, \qquad l = 0, 1, 2, \cdots,$$

such that $\{y_j^l, \ j = 0, 1, \cdots, m\} \subset S_1(x_l)$ and $y_j^l \to y_j$ as $l \to \infty$. For l large, $d(y_j^l, y_{j+1}^l) < \frac{1}{4}$; we can connect y_j^l and y_{j+1}^l by a minimal geodesic in M_l. This implies that for l large we can connect y^l and z^l by broken geodesics with total length $\leq \frac{1}{4}m$. Clearly, for each l, this curve lies in $B_{3/2}(x_l) - B_{1/2}(x_l)$ and we can radially project this curve to $S_1(x_l)$. By (b), the length of the projection curve is bounded by $c(H, i_0, n) \cdot \frac{1}{4}m$. This can apply to any subsequence of the sequence of manifolds, and proves (a) (see III of [13]).

5. The curvature estimates and removability of singularities of the metric g'

In this section, we will prove that the singular points of the Einstein metric g' can be removed, and that g' can be extended to an Einstein metric on a compact manifold M.

To start, let M_1 be obtained from M' by adding $\{v_1, \cdots, v_{s+1}\}$ to M' with the topology induced by the distance function d of g' on M'.

Clearly, d can be extended to M_1, and $M_1 - \{v_1, \cdots, v_{s+1}\}$ is isometric to M' with Riemannian metric g'.

Let us fix $v = v_i$, $1 \le i \le s+1$, and let r be the distance function from v, i.e., $r(x) = d(x, v)$ for $x \in M_1$. We take N to be a neighborhood of v in M_1 such that $N \cap \{v_1, \cdots, v_{s+1}\} = v$. We may assume that $N \supset B(v, 2\rho_0) = \{x \in M_1 | r(x) < 2\rho_0\}$ for some $\rho_0 > 0$. Let F be the curvature tensor for the metric g', and let R define a symmetric operator $\Lambda^2(M') \to \Lambda^2(M')$ on the space of two-forms. We take N small such that N is orientable. (In fact, we may assume that N is simply connected.) The Hodge star operator $*\colon \Lambda^2 \to \Lambda^2$ satisfies $*^2 = +1$, and the bundle Λ^2 splits into a direct sum

$$\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-.$$

Here Λ^2_+, Λ^2_- are called the bundles of self-dual and anti-self-dual 2-forms respectively.

For the Riemannian metric g' on N, we consider the bundle Λ^2_+ (or Λ^2_- with the induced Riemannian connection D_+ (or D_-). The Riemannian curvature tensor $F\colon \Lambda^2 \to \Lambda^2$ is given by

$$R(e_i \wedge e_j) = \tfrac{1}{2} \sum R_{ijkl} e_k \wedge e_l,$$

where $\{e_i\}$ is a local orthonormal basis of l-forms. We can write R as a block matrix relative to the decomposition $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$:

$$R = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix},$$

where $B \in \mathrm{Hom}(\Lambda^2_-, \Lambda^2_+)$, and $A \in \mathrm{End}\,\Lambda^2_+$ and $c \in \mathrm{End}\,\Lambda^2_-$ are self-adjoint. For the Einstein metric g', $B \equiv 0$. The curvature Ω_+ (resp. Ω_-) is self-dual (resp. anti-self-dual), and $|R|^2 = |\Omega_+|^2 + |\Omega_-|^2$ [2].

Let us fix $F = \Omega_+$ (or Ω_-) and $f = |F|$. We have a Weitzenböck formula for self-dual (or anti-self-dual) 2-forms [11]

$$2D_+ D_+^* = \nabla^* \nabla - 2W^+(\cdot) + s/3 + [P_+ F, \cdot],$$

where P_+ is the projection onto self-dual 2-forms, ∇ is the induced connection on $T^*N \otimes \Lambda^2_+$, and W and s are the Weyl tensor and scalar curvature of the metric g'. Applying this Weitzenböck formula to F and using $D_+^* F \equiv 0$, we have

$$\Delta F = s/3 \cdot F - 2W^+(F) + [P_+ F, F]. \tag{5.1}$$

Noting that $|W^+| \le |A| = |F|$ and $|P_+ F| \le |F|$, this implies that for an absolute constant $c > 0$,

$$\Delta f \ge \tfrac{s}{3} f - c f^2. \tag{5.2}$$

The following proposition gives the basic estimate of the curvature F on N.

Proposition 5.3. *We have $f \le \varepsilon(r)/r^2$ where $\varepsilon(r)$ is a decreasing function and $\lim_{r\to 0} \varepsilon(r) = 0$.*

Proof. For each small $\rho > 0$, we define the metric $\overline{g} = g'/\rho^2$. Since the Yang-Mills equations and the L^2 norm of the curvature F of the fixed connection D_+ are conformally invariant, we have $\overline{f} = |F|_{\overline{g}} = \rho^2 f$. We fix the point $x_0 \in N$ such that $r(x_0) = \rho$. From the above sections, g' is the weak limit. Since

$$\frac{1}{8\pi^2}\int_N f^2\, dv' \le \frac{1}{8\pi^2}\int_N |R|^2\, dv' \le \frac{1}{8\pi^2}\lim_{k\to\infty}\int_M |R(g^k)|^2\, dv_k = \chi(M),$$

we have

$$\int_N \overline{f}^2\, d\overline{g} \le 8\pi^2 \chi(M).$$

We now observe the uniform Sobolev inequality for the manifold M in $G(M)$. For $(M, g) \in G(M)$ and $\phi \in H_1^2(M)$,

$$\left(\int_M \phi^4\, dg\right)^{1/2} \le c\left[\int_M |\nabla\phi|^2\, dg + \int_M \phi^2\, dg\right].$$

By taking limits, Proposition 3.10 then implies that for any $\phi \in H_1^2(N)$ with compact support in N

$$\left(\int_N \phi^4\, dg'\right)^{1/2} \le c(i_0)\left[\int_N |\nabla\phi|^2\, dg' + \int_N \phi^2\, dg'\right].$$

Taking ρ small, $\rho \le 1$, we obtain the following Sobolev inequality on N:

$$\left(\int_N \phi^4\, d\overline{g}\right)^{1/2} \le c(i_0)\int_N |\overline{\nabla}\phi|^2\, d\overline{g} + \int_N \phi^2\, d\overline{g}. \tag{5.4}$$

By taking N small, we have

$$\int_N |F|_{\overline{g}}^2\, d\overline{g} = \int_N |F|_g^2\, dg$$

small. Using (5.2) for the metric $\overline{g}$ and noting that the scalar curvature $\overline{s}$ of $\overline{g}$ satisfies $\overline{s} \ge -12$, by (5.4) and Moser iteration (see §4), we easily obtain

$$(5.5) \qquad \sup_{B_{\overline{g}}(x_0, 1/2)} \overline{f} \le c(i_0, d) \left(\int_{B_{\overline{g}}(x_0, 3/4)} |F|^2_{\overline{g}} d\overline{g} \right)^{1/2},$$

which clearly implies (5.3).

Now, for $\rho > 0$ small, since $\mathrm{inj}(g^k) \ge i_0 > 0$, a result of Croke [9] implies that $\mathrm{Vol}(B_{gk}(x, \frac{1}{2}\rho)) \ge c\rho^4$ for a constant c independent of ρ. By taking limits, we have

$$\mathrm{Vol}(B_{g'}(x, \tfrac{1}{2}r(x))) \ge cr(x)^4,$$

and for $\rho = r(x_0)$, $x_0 \in N$, and ρ small, we obtain

$$\mathrm{Vol}(B_{\overline{g}}(x_0, \tfrac{1}{2})) \ge c.$$

Using (5.3), we have the bound for the curvature tensor of $\overline{g}$ on $B_{\overline{g}}(x_0, \frac{1}{2})$, $|R_m(\overline{g})| \le c$; these and a local injectivity radius estimate easily give the following theorem.

Theorem 5.6. *Let* $\mathrm{inj}(v)$ *denote the injectivity radius of* M' *at* v. *Then* $\mathrm{inj}(v) \ge cr(v)$ *for a constant* c *independent of* $r(v)$.

Let

$$A(\rho, \overline{\rho}) = B_\rho(v) - B_{\overline{\rho}}\{v\} = \{x \in M, \overline{\rho} < r(x) < 2\rho\};$$

here $A(\rho, \overline{\rho})$ is an open subset of M'. We consider $A(p, \overline{\rho})$ as a Riemannian manifold with metric g'. Let $\frac{1}{\rho}A(\rho, \overline{\rho})$ be the Riemannian manifold $A(\rho, \overline{\rho})$ with metric g'/ρ^2. For any fixed large number K, by Proposition 5.3 we have

$$(5.7) \qquad |R\rho| = \rho^2|R| \le \frac{2\varepsilon(2\rho)}{(\rho/K)^2} \cdot \rho^2 \le 2K^2\varepsilon(2\rho) \quad \text{on } \frac{1}{\rho}A\left(\rho, \frac{\rho}{K}\right),$$

where R_ρ denotes the curvature tensor of the metric g'/ρ^2. Let $x_\rho \in \frac{1}{\rho}A(\rho, 0)$, such that $r(x_\rho) = \rho$. Then $\{(\frac{1}{\rho}A(\rho, 0), x_\rho)\}$ is a family of Einstein manifolds, and for fixed K, $|R_\rho|$ converges to 0 uniformly when $\rho \to 0$ on $\frac{1}{\rho}A(\rho, \rho/K)$. We now are ready to prove

Theorem 5.8. $(\frac{1}{\rho}A(\rho, 0), x_\rho)$ *converges to* $(D^4 - \{0\}, \vec{e})$ *weakly in* c^4 *topology, where* $D^4 = \{w \in R^4, |w| < 2\}$ *is the Euclidean ball of radius* 2.

Proof. The proof is similar to that of Theorem 3.1. For fixed K, we show that $(\frac{1}{\rho}\overline{A(\rho, \rho/K)}, x\rho)$ converges to a flat manifold $\overline{D(1/K)}$.

Using (5.6) and (5.7), as in the proof of Proposition 3.2, a subsequence of $(\frac{1}{\rho}\overline{A(\rho, \rho/K)}, x_\rho)$ converges to a flat manifold $\overline{D(1/K)}$.

In order to show that

$$(\tfrac{1}{\rho}A(\rho, 0), x_\rho) \xrightarrow{c^4} D^4 - \{0\},$$

we need only show that for each subsequence

$$\left\{\frac{1}{\rho_k}A(\rho_k, 0), x_{\rho_k}\right\}$$

of $\{\frac{1}{\rho}A(\rho, 0), x_\rho\}$, there exists a subsequence of

$$\left\{\frac{1}{\rho_k}A(\rho_k,), x_{\rho_k}\right\}$$

which converges to $D^4 - \{0\}$ in c^4 topology.

For such a subsequence, the Einstein manifold

$$\left\{\frac{1}{1\rho_k}A\left(\rho_k, \frac{\rho_k}{K}\right), x_{\rho_k}\right\}$$

has a subsequence which converges to a flat manifold $(\overline{D(\frac{1}{K})}, \rho_K)$. By diagonalization (passing to a subsequence if necessary), we may assume that for each positive integer K, we have

$$\left(\frac{1}{\rho_k}A\overline{\left(\rho_k, \frac{\rho_k}{K}\right)}, x_{\rho_k}\right) \xrightarrow{c_4} \left(\overline{D^4\left(\frac{1}{K}\right)}, e_n\right).$$

This implies

$$\left(\frac{1}{\rho_k}A(\rho_k, 0), x_{\rho_k}\right) \overset{c^4 \text{ weakly}}{\longrightarrow} (D^4(0), e),$$

where $\overline{D^4(0)} = \bigcup_n \overline{D^4(\frac{1}{K})}$, and clearly $(\overline{D^4(\frac{1}{K})}, e_K)$ is isometrically embedded in

$$\left(\overline{D^4\left(\frac{1}{K+1}\right)}, e_{K+1}\right)$$

as a submanifold. It is easy to see by 3.25 that $\overline{D^4(0)}$ has a one-point compactification $\overline{D^4}$, such that $\overline{D^4} = \overline{D^4(0)} \cup \{0\}$ and that the distance function d on $\overline{D^4(0)}$ extends to $\overline{D^4}$. $D^4(0)$ is a flat Riemannian manifold; we claim that $\overline{D^4}$ is isometric to the standard Euclidean ball $\{w \in R^4, |w| \le 2\}$. In fact, if $\overline{D^4(0)}$ is simply connected, then it is not difficult to prove that, using a holonomy argument, we can isometrically embed $\overline{D^4(0)}$ into R^4. We can extend the flat metric smoothly on $\overline{D^4}$, and $\overline{D^4}$ is the standard Euclidean ball of radius 2 in R^4. If $A(\rho, \overline{\rho})$ is simply connected, then $D^4(\frac{1}{K})$ is simply connected, which implies that $\overline{D^4(0)}$ is simply connected, so in order to finish the proof of Theorem 5.8, we only need to prove the following lemma.

Lemma 5.9. *$A(\rho, \overline{\rho})$ is simply connected for small $0 < \overline{\rho} < \rho$.*

Proof. Let α be a closed curve in $A(\rho, \overline{\rho})$. Since α is compact, there exists $\eta > 0$ such that $\overline{\rho} + \eta < r|_\alpha < \rho - \eta$. Using Theorem 3.11, we take r_l small such that $A(\rho, \overline{\rho}) \subset D(r_l)$. Note that $f^k(r_l)^* g^k \to g'$ in c^2 topology. Let

$$A_k(\rho, \overline{\rho}) = f^k(r_l)^{-1}(A(\rho, \overline{\rho})).$$

Then $A_k(\rho, \overline{\rho}) \subset F_k(r_l)$. From 3.2, 3.10 and 3.11, it is easily seen that for l and k sufficiently large, and δ small, we have

$$A_k(\rho - \eta, \overline{\rho} + \eta) \subset B^k_{\rho-\eta+\delta}(m^k_l) - aB^k_{\overline{\rho}+\eta-\delta}(m^k_l) \subset A_k(\rho, \overline{\rho}),$$

for $m^k_l \in M^k$, thus α can be contracted to a point in $A(\rho, \overline{\rho})$. This completes the proofs of Lemma 5.9 and Theorem 5.8.

We deform the metric g' conformally to the metric $\tilde{g} = g'/r^2$ on N. By the conformal invariance of the Yang-Mills equation and L^2-norm of the curvature F of the fixed connection D_+, for $\tilde{f} = |F|_{\tilde{g}} = r^2 f$, we obtain the Sobolev inequality for the metric $\tilde{g}$ on N:

$$\left(\int_N \phi^4 \, d\tilde{g}\right)^{1/2} \le c\left[\int_N |\tilde{\nabla}\phi|^2 \, d\tilde{g} + \int_N \phi^2 \, d\tilde{g}\right];$$

here ϕ has compact support in N.

Let $\tau = -\log r$ be the distance function of $\tilde{g}$ to the point v. We are now ready to give the decay estimate of $\tilde{f}$.

Theorem 5.10. *There exist a constant $c > 0$, and $\overline{\delta} > 0$, such that for τ_0 large, we have*

$$\tilde{f}(\tau) \le c\tilde{f}(\tau_0)e^{-\overline{\delta}(\tau-\tau_0)} \quad \textit{for } \tau \ge \tau_0.$$

Remark. Theorem 5.10 implies that $f(r) \le c/r^{2-\overline{\delta}}$.

Proof. Let $h^2 : D^4 \to R$ be the square of the distance function on D^4 at the origin, i.e., $h^2(w) = |w|^2$, $w \in D^4$. We take $T \ge 10$ large and fix it, and set $K = e^{4T}$. By using Theorem 5.8, there are diffeomorphisms Ψ_ρ from $(\frac{1}{\rho}A(\frac{3}{4}\rho, \rho/2K), x_\rho)$ into (D^4, e) such that $(\Psi_\rho^{-1})^*(g'/\rho^2)$ converges to the flat Euclidean metric ds^2 on D^4, and for $S(\rho) = \{y \in M, r(y) = \rho\}$, $\Psi_\rho(\frac{1}{\rho}S(\rho))$ converges to the standard sphere $S_0(1) = \{w \in D^4, |w| = 1\}$ of radius 1 in D^4. This implies that for a small $\rho_0 > 0$, if $\rho \le \rho_0$, we have

$$(1-\delta)\frac{r^2}{\rho^2} \le \Psi_\rho^*(h^2) \le (1+\delta)\frac{r^2}{\rho^2}$$

for small $\delta > 0$ on $\overline{\frac{1}{\rho}A(\frac{1}{2}\rho, \rho/2K)}$, and

$$\left|(\Psi_\rho^{-1})^*\left(\frac{1}{\rho^2}g'\right) - ds^2\right|_{C^1} < \delta.$$

Let Δ_ρ and Δ_0 be the Laplacians on $\frac{1}{\rho}A(\rho, 0)$ and D^4 respectively. Taking δ small, then $\Delta_0 h^2 = 8$ on D^4 implies that $\Delta_\rho(\Psi_\rho)^*(h^2) \ge 7\frac{1}{2}$, and that $|\nabla_\rho(\Psi_\rho)^* h^2|^2 \le 1 + \frac{1}{36}$. Let

$$S_\rho^2 = (\Psi_\rho)^*(h^2), \quad \tilde{g}_\rho = \frac{1}{\rho^2 S_\rho^2}g', \quad \tilde{f}_\rho = |F|_{\tilde{g}_\rho}.$$

Let $\tilde{s}_\rho$ be the scalar curvature of the metric $\tilde{g}_\rho$. Then we have

$$\begin{aligned}\tilde{s}_\rho &= (-6\Delta_\rho S_\rho^{-1} + 12\rho^2 \cdot S_\rho^{-1})S_\rho^3 \\ &= 3\Delta_\rho(S_\rho^2) - 18|\nabla_\rho S_\rho|^2 + 12\rho^2 S_\rho^2 \ge 4.\end{aligned}$$

Lemma 5.11. *Let*

$$F(\tau_1, \tau_2) = \int_{\tau_1 \le \tau \le t_2} \tilde{f}^2.$$

Then for δ small and T large, we have

$$\begin{aligned}F(\tau_1, \tau_2) \le\ & 2^{(\tau_1-\tau_\rho)/-2(1+\delta)}F(\tau_\rho, \tau_1) \\ &+ \left(\frac{4}{T} + 2e^{(\tau_1-\tau_\rho)/-2(1+\delta)}\right)F(\tau_2, \tau_2 + T(1+3\delta)),\end{aligned}$$

where

$$\tfrac{1}{2}T(1-\delta) + \tau_0 \le \tau_1 \le \tau_\rho + T(1+\delta) \le \tau_2 = \tau_\rho + T(1-\delta).$$

Proof. We take $\tau_1 \ge \tau_0 = -\log\rho_0$, and a function $\phi_\varepsilon\colon (-\infty, \infty) \to [0, 1]$ such that $\phi_\varepsilon \equiv 1$ on $[\tau+\varepsilon, T+\tau]$, $\phi_\varepsilon \equiv 0$ on $(-\infty, \infty) - [\tau, 2T+\tau]$, and that ϕ_ε is linear on $[\tau, \tau+\varepsilon]$ and $[T+\tau, 2T+\tau]$. Then $\phi_\varepsilon' = \frac{1}{\varepsilon}$ on $[\tau, \tau+\varepsilon]$, and $\phi_\varepsilon' = -\frac{1}{T}$ on $[T+\tau, 2T+\tau]$. Let $u_\varepsilon = \phi_\varepsilon(r_\rho + \tau)$, where r_ρ is the distance function from $\frac{1}{\rho}S(\rho)$ of the metric $\tilde{g}_\rho$. Then $|\nabla_\rho u_\varepsilon|^2 \le |\phi_\varepsilon'|^2$. Using Proposition 5.3, and taking ρ_0 small, we have

$$\begin{aligned}\tilde{f}_\rho &= \rho^2 S_\rho^2 f \le \rho^2(1+\delta)\frac{r^2}{\rho^2}f \\ &\le (1+\delta)\tilde{f} \le \frac{1}{3c} \quad \text{on } A\left(\frac{3}{4}\rho, \frac{\rho}{2K}\right), \text{ for } \rho < \rho_0.\end{aligned}$$

Applying the Weitzenböck formula (5.2) yields

$$\tilde{\Delta}_\rho \tilde{f}_\rho \geq \frac{\overline{s}_\rho}{3}\tilde{f}_\rho - c\tilde{f}_\rho^2 \geq \frac{4}{3}\tilde{f}_\rho - c\cdot\frac{1}{3c}\tilde{f}_\rho \geq \tilde{f}_\rho,$$

where $\tilde{\Delta}_\rho$ is the Laplacian of the metric $\tilde{g}_\rho$. Using this inequality and integrating by parts, we derive

$$\begin{aligned}\int u_\varepsilon^2|\tilde{\nabla}_\rho\tilde{f}_\rho|^2 &\leq -\int u_\varepsilon^2\tilde{\Delta}_\rho\tilde{f}_\rho\cdot\tilde{f}_\rho - 2\int u_\varepsilon\langle\tilde{\nabla}_\rho u_\varepsilon, \tilde{\nabla}_\rho\tilde{f}_\rho\rangle\tilde{f}_\rho \\ &\leq -\int u_\varepsilon^2\tilde{f}_\rho^2 - 2\int u_\varepsilon\langle\tilde{\nabla}_\rho u_\varepsilon, \tilde{\nabla}_\rho\tilde{f}_\rho\rangle\tilde{f}_\rho.\end{aligned}$$

Applying the Cauchy inequality to the last term on the right, we obtain

$$\int u_\varepsilon^2|\tilde{\nabla}_\rho\tilde{f}_\rho|^2 + \int u_\varepsilon^2\tilde{f}_\rho^2 \leq \tfrac{1}{2}\int u_\varepsilon^2|\tilde{\nabla}_\rho\tilde{f}_\rho|^2 + 2\int|\tilde{\nabla}_\rho u_\varepsilon|\tilde{f}_\rho^2.$$

Hence,

$$\int u_\varepsilon^2\tilde{f}_\rho^2 \leq \frac{2}{\varepsilon}\int_{0\leq r_\rho\leq\varepsilon}\tilde{f}_\rho + \frac{2}{T}\int_{T\leq r_\rho\leq 2T}\tilde{f}_\rho^2.$$

Letting $\varepsilon \rightarrow 0$, and $E(t) = \int_{t\leq r_\rho} u_0^2\tilde{f}_\rho^2$, we have

$$E(t) \leq -2\frac{d}{dt}E(t) + \frac{2}{T}\int_{T\leq r_\rho\leq 2T}\tilde{f}_\rho^2, \qquad t\leq T.$$

Solving the differential inequality gives

$$(*) \qquad E(t) \leq E(0)e^{-t/2} + \frac{2}{T}\int_{T\leq r_\rho\leq 2T}\tilde{f}_\rho^2.$$

On the other hand, $(1-\delta)r^2 \leq \rho^2 S_\rho^2 \leq (1+\delta)r^2$ implies that $(1+\delta)^{-1}\tilde{g} \leq \tilde{g}_\rho \leq (1-\delta)^{-1}\tilde{g}$ on $\frac{1}{\rho}A(\frac{1}{2}\rho, \frac{1}{2}\rho K)$. We also have

$$r_\rho \leq \frac{\tau-\tau_\rho}{(1-\delta)} \quad\text{and}\quad \frac{\tau-\tau_\rho}{(1+\delta)} \leq r_\rho,$$

where $\tau_\rho = -\log\rho = \tau(\rho)$. Combining these with $(*)$ yields

$$\begin{aligned}&\int_{(1+\delta)t\leq\tau-\tau\rho\leq T(1-\delta)}\tilde{f}^2 \\ &\leq \int_{t\leq r_\rho\leq T}\tilde{f}_\rho^2 \leq E(t) \\ &\leq e^{-t/2}\int_{\tau_\rho\leq\tau\leq\tau_\rho+2T(1+\delta)}\tilde{f}^2 + \frac{2}{T}\int_{(1-\delta)T\leq\tau-\tau_\rho\leq 2(1+\delta)T}\tilde{f}^2\end{aligned}$$

for $0 \le t \le T$. Let $F(\tau_1, \tau_2) = \int_{\tau_1 \le \tau \le \tau_2} \tilde{f}^2$. Thus

$$F(\tau_1, \tau_2) \le e^{-(\tau_1 - \tau_\rho)/2(1+\delta)} F(\tau_\rho, \tau_2 + T(1+3\delta)) + \frac{2}{T} F(\tau_2, \tau_2 + T(1+3\delta)),$$

where

$$\tau_\rho \le \tau_1 \le \tau_\rho + (1+\delta)T \le \tau_2 = \tau_\rho + T(1-\delta).$$

By taking δ small and T large such that $e^{-(1-\delta)/(1+\delta)\cdot T/2} < \frac{1}{4}$, when $\tau_1 \ge \tau_\rho + \frac{1}{2}(1-\delta)T$, we have

$$\begin{aligned} F(\tau_1, \tau_2) \le{} & e^{-(\tau_1 - \tau_\rho)/2(1+\delta)} F(\tau_\rho, \tau_1) + \frac{1}{2} F(\tau_1, \tau_2) \\ & + e^{-(\tau_1 - \tau_\rho)/(1+\delta)} F(\tau_2, \tau_2 + T(1+3\delta)) \\ & + \frac{2}{T} F(\tau_2, \tau_2 + T(1+3\delta)), \end{aligned}$$

which clearly implies Lemma 5.11.

We are now ready to finish the proof of Theorem 5.10. Since Lemma 5.11 is true for all $\tau_\rho \ge \tau_{\rho_0}$, we will use Lemma 5.11 to iterate. To start, taking $\tau_1 = \tau_\rho + \frac{1}{2}(1-\delta)T$ in Lemma 5.11, we have

$$\begin{aligned} & F(\tau_\rho + \tfrac{1}{2}(1-\delta)T, \tau_\rho + (1-\delta)T) \\ & \quad \le 2e^{-((1-\delta)/4(1+\delta))} F(\tau_\rho, \tau_\rho + \tfrac{1}{2}(1-\delta)T) \\ & \quad\quad + \varepsilon F(\tau_\rho + (1-\delta)T, \tau_\rho + \tfrac{3}{2}T(1+2\delta)), \end{aligned}$$

where $\varepsilon = 4/T + 2e^{-((1-\delta)/4(1+\delta))T}$. We take δ small such that $\frac{3}{2}(1+2\delta) < 2(1-\delta)$, which implies that

$$\begin{aligned} & F\left(\tau_\rho + \frac{1}{2}(1-\delta)T, \tau_\rho + (1-\delta)T\right) \\ & \quad \le 2e^{-\delta T} F\left(\tau_\rho, \tau_\rho + \frac{1}{2}(1-\delta)T\right) \\ & \quad\quad + \varepsilon F(\tau_\rho + (1-\delta)T, \tau_\rho + 2(1-\delta)T), \qquad \delta \le \frac{1-\delta}{4(1+\delta)}. \end{aligned} \tag{5.12}$$

Hence we set $\tau_{\rho_m} = \tau_\rho + \frac{m}{2}(1-\delta)T$, $m = 0, 1, 2, \cdots$, so that by inequality (5.12) we have

$$\begin{aligned}F(\tau_{\rho_{m+1}}, \tau_{\rho_{m+2}}) &= F(\tau_{\rho_m} + \tfrac{1}{2}(1-\delta)T, \tau_{\rho_m} + (1-\delta)T) \\ &\le 2e^{-\delta T} F(\tau_{\rho_m}, \tau_{\rho_m} + \tfrac{1}{2}(1-\delta)T) \\ &\quad + \varepsilon F(\tau_{\rho_m} + (1-\delta)T, \tau_{\rho_m} + 2(1-\delta)T) \\ &\le 2e^{-\delta T} F(\tau_{\rho_m}, \tau_{\rho_{m+1}}) + \varepsilon F(\tau_{\rho_{m+2}}, \tau_{\rho_{m+4}}).\end{aligned}$$

Summing up from m to infinity leads to the inequality

$$\begin{aligned}F(\tau_{\rho_{m+1}}, \infty) &\le (2e^{-\delta T}) F(\tau_{\rho_m}, \infty) + 2\varepsilon F(\tau_{\rho_{m+2}}, \infty) \\ &\le (2e^{-\delta T}) F(\tau_{\rho_m}, \infty) + 2\varepsilon F(\tau_{\rho_{m+1}}, \infty).\end{aligned}$$

For T large and δ small, we may assume that $2\varepsilon \le \frac{1}{2}$, and $4e^{-\delta T/2} \le \frac{1}{2}$, so that we have

$$\begin{aligned}F(\tau_{\rho_{m+1}}, \infty) &\le (4e^{-\delta T}) F(\tau_{\rho_m}, \infty) \\ &\le (e^{-\delta T/2}) F(\tau_{\rho_m}, \infty) \le (e^{-\delta T/2})^{m+1} F(\tau_\rho, \infty) \\ &\le e^{-\delta(m+1)T/2} F(\tau_\rho, \infty) \\ &\le e^{-\delta(\tau_{\rho_{m+1}} - \tau_\rho)/(1-\delta)} F(\tau_\rho, \infty).\end{aligned} \tag{5.13}$$

By noting that $\tau_\rho \ge \tau_{\rho_0}$ is arbitrary, inequality (5.13) and (5.5) clearly imply Theorem 5.10.

We are now ready to prove the main theorem of this section.

Theorem 5.14. *The norm $|R|$ of the curvature tensor R of the metric g' is bounded.*

We start with the following lemma.

Lemma 5.15. *There exists $p > 2$, such that, for small $r_0 > 0$,*

$$\int_{B_{r_0}(v)} |R|^p \, dg' \le c.$$

Proof. By Theorem 5.8,

$$\tfrac{1}{r} S(r) = \tfrac{1}{r}\{y \in N, r(y) = r\} \subset N$$

converges to the standard sphere $S^4(1)$ of radius 1 in R^4. In the Hausdorff metric, we take r_0 small such that we have the estimates of Hausdorff measure of $\frac{1}{r}S(r)$ for $r \le r_0[F]$:

$$H^3(\tfrac{1}{r}S(r)) \le 2H^3(S^4(1)) = w\omega_4 \cdot 4 = 8\omega_4,$$

which implies

$$H^3(S(r)) \le 8\omega_4 r^3.$$

Hence, taking $2 < p < 4/(2-\delta)$, by Theorem 5.10, we have

$$\int_{B_{r_0}(\nu)} |R|^p \, dg' \le \int_{B_{r_0}(\nu)} \left(\frac{cr^{\bar{\delta}}}{r^2}\right)^p dg'$$

$$\le \int_0^{r_0} \left(\int_{S(r)} \frac{c}{r^{2p-\bar{\delta}_p}} \, d\omega \right) dr \le c r_0^{r-2p+\bar{\delta}p},$$

which completes the proof of Lemma 5.15.

Now Theorem 5.14 follows from $\Delta|R| \ge -c|R|^2$ and a method of Serrin [24] by iteration.

We have shown that the curvature of M' is bounded and that M' is a weak limit of M_k with $\mathrm{inj}(M_k) \ge i_0 > 0$. We now prove that the exponential map of M_1 at v can be defined.

Lemma 5.16. *The exponential map of M_1 at v can be defined, and it is a homeomorphism between a small Euclidean ball and a small geodesic ball of M_1 at v.*

Proof. All we need to show is that for two different geodesic segments γ and γ_1 of M_1 from v, i.e., for $\gamma(1) - \gamma_1(1) = v$, $\gamma(t), \gamma_1(t) \subset M'$ for $0 \le t < 1$, and $\|\gamma\| < i_0/100$, $\|\gamma_1\| < i_0/100$, we have $\gamma(0) \ne \gamma_1(0)$.

Suppose that $\gamma(0) = \gamma_1(0)$. First, for each $t < 1$, $\gamma|[0, t]$ and $\gamma_1|[0, t]$ are limits of sequences of segments (α_k) and (β_k), where $\alpha_k, \beta_k \subset M_k$ and $\alpha_k(0) = \beta_k(0)$. This follows from the fact that $M_k \to M'$ weakly in C^2-norm. For k large, α_k and β_k will be contained in the geodesic ball of M_k with curvature bounded independent of k. The Rauch comparison theorem then implies that $d(\alpha_k(t), \beta_k(t)) \ge C > 0$ for $t \ge 1/2$, and therefore that $d(\gamma(t), \gamma_1(t)) \ge C > 0$; this contradicts that $\gamma(1) = \gamma_1(1)$.

Clearly, the same argument also implies

Lemma 5.17. *For any $x \in M_1 - \{v\}$ and $r(x) < i_0/2$, we have* $\mathrm{inj}(x) \ge r(x)$.

The following theorem is a special case of the removable singularity theorem of Smith and Yang [26].

Theorem 5.18. *If M is a Riemannian manifold with isolated singular point v, and $M' = M - \{v\}$, such that*

(a) *the exponential map of M at v is defined and is a homomeorphism between a small Euclidean ball and a small geodesic ball of M at v,*

(b) $(1/\rho)A(\rho, 0)$ *converges weakly to $D^n - (0)$ in C^2-norm,*

(c) *the curvature of M is bounded, and*

(d) $\operatorname{inj}(x) \geq r(x)$ *for* $x \in M'$ *and* $r(x)$ *small,*
then there exists a harmonic coordinate on M' *around* v *such that in this coordinate system, the metrics* g *of* M' *can be extended to a* $C^{1,\alpha}$ *metric* g *on* M.

Proof. In our case, it is easily seen that we still can define the almost linear coordinate of M at v as in [19], and the estimates of the almost linear coordinate still hold, so we can use the almost linear coordinate to define a harmonic coordinate. Note that all the estimates of [19] can be done away from the point v. We have

$$\|g'\|_{C^{1,\alpha}} + \|g'^{-1}\|_{C^{1,\alpha}} \leq c$$

in the harmonic coordinate, and g' can be extended to a $C^{1,\alpha}$-metric g on M.

Theorem 5.19. *The Einstein metric* g' *on* M' *can extend to a smooth Einstein metric* g *on* M.

Proof. Using 5.17, there exists a harmonic coordinate neighborhood N of v, with harmonic coordinate (x^i) on N, such that g' is $C^{1,\alpha}$ on $(N-(v))$ so that g' can be extended to a $C^{1,\alpha}$-metric g on N. We then have

$$\Delta g^{ij} = 2\sum g^{rs} g^{pq} \Gamma^i_{pr} \Gamma^j_{qs} + 2\sigma g^{ij}$$

in the weak sense on N. The regularity theory of the elliptic equation then implies that g^{ij} is smooth on N, and that g is Einstein on N.

References

[1] J. W. Alexander, *The combinatorial theory of complexes*, Ann. of Math. (2) **31** (1930) 292–320.

[2] N. F. Atiyah, N. J. Hitchin & I. M. Singer, *Self-duality in four-dimensional Riemannian geometry*, Proc. Roy. Soc. London Ser. A **362** (1978) 425–461.

[3] M. Berger, *Une borne inferieure pour le volume d'une variete riemannienne en function du rayon d'injective*, Ann. Inst. Fourier (Grenoble) **30** (1980) 259–265.

[4] M. Berger & D. Ebin, *Some decompositions of the space of symmetric tensors on a Riemannian manifold*, J. Differential Geometry **3** (1969) 379–392.

[5] A. Besse, *Geometric Riemannienne en Dimension* 4, CEDIC, Paris, 1981.

[6] ——, *Einstein manifolds*, Springer, New York, 1986.

[7] R. L. Bishop & R. J. Crittenden, *Geometry of manifolds*, Academic Press, New York, 1964.

[8] P. Buser & H. Karcher, *Gromov's almost flat manifolds*, Astérisque, No. 81, Soc. Math. France, Paris, 1981.

[9] C. B. Croke, *Some isoperimetric inequalities and eigenvalue estimates*, Ann. Sci. École Norm. Sup., Paris **13** (1980) 419–435.

[10] P. Ehrlich, *Metric deformations of curvature*, Geometriae Dedicata **5** (1976) 1–23.

[11] H. Federer, *Geometric measure theory*, Springer, Berlin, 1969.

[12] D. S. Freed & K. K. Uhlenbeck, *Instantonprint Einstein s and four-manifolds*, Springer, Berlin, 1984.

[13] L. Z. Gao, $L^{n/2}$*-curvature pinching*, preprint.
[14] D. Gilbarg & N. Trudinger, *Elliptic partial differential equations of second order*, Springer, Berlin, 1984.
[15] R. E. Greene & H. Wu, *Lipschitz convergence of riemannian manifolds*, preprint.
[16] M. Gromov, *Structures metriquee pour les varietes Riemanniennes*, CEDIC, Paris, 1981.
[17] R. S. Hamilton, *Three-manifolds with positive ricci curvature*, J. Differential Geometry **17** (1982) 255–306.
[18] S. Ilias, *Constantes explicites pour ies inegalites de sobolev sur les varieties riemanniennes compactes*, Ann. Inst. Fourier (Grenoble) **33** (1983) 151–165.
[19] J. Jost, *Harmonic mapping between Riemannian manifolds*, Australian National University, 1986.
[20] H. Karcher, *Riemannian center of mass and mollifier smoothing*, Comm. Pure Appl. Math. **30** (1977) 509–541.
[21] S. Kobayashi & K. Nomizu, *Foundations of differential geometry*, Vol. I, Wiley, New York, 1963.
[22] J. Munkres, *Topology*, Prentice-Hall, Englewood Cliffs, NJ, 1975.
[23] S. Peters, *Convergence of Riemannian manifolds*, Compositio Math. **62** (1987) 3–16.
[24] J. Sacks & K. K. Uhlenbeck, *The existence of minimal immersions of 2-spheres*, Ann. of Math. (2) **113** (1981) 1–24.
[25] J. Serrin, *Local behavior of solutions of quasi-linear equations*, Acta Math. **3** (1964) 247–302.
[26] P. Smith & D. Yang, *Removing point singularities of Riemannian metrics*, preprint.
[27] E. H. Spanier, *Algebraic topology*, Springer, Berlin, 1966.
[28] K. K. Uhlenbeck, *Removable singularities in Yang-Mills fields*, Comm. Math. Phys. **83** (1982) 11–29.

RICE UNIVERSITY